PG – 118

IV Semester M.Sc. Degree Examination, June 2015
(RNS)
MATHEMATICS
M – 403 – C : Theory of Numbers

Time : 3 Hours Max. Marks : 80

Instructions : *Answer **any five full** questions.*
__All__ questions carry __equal__ marks.

1. a) Define Euler's totient function φ. Prove that $\sum_{d|n} \varphi(d) = n$, for $n \geq 1$. 6

   b) If $n \geq 1$, prove that $\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$. 6

   c) Find all integers n such that $\varphi(n) = \varphi(2n)$. 4

2. a) Prove that $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$, if $n \geq 1$. 6

   b) Define the Mangoldt function $\wedge$. If $n \geq 1$, prove that

      i) $\sum_{d|n} \wedge(d) = \log n$, 6

      ii) $\wedge(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}$ 4

      $= -\sum_{d|n} \mu(d) \log d$.

3. a) Define a multiplicative function. If f and g are multiplicative, prove that their Dirichlet product $f * g$ is also multiplicative. 4

   b) Let f be multiplicative. Prove that :

      i) f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$, for all $n \geq 1$. 4

      ii) $\sum_{d|n} \mu(d) f(d) = \prod_{p|n} (1 - f(p))$. 4

   c) Obtain the Bell series for :

      i) Euler's totient function φ.

      ii) Liouville's function λ. 4

4. a) If $(a, m) = 1$, prove that $a^{\varphi(m)} \equiv 1 \pmod m$. 5

   b) Given a prime p, let $f(x) = c_0 + c_1x + \ldots + c_nx^n$, be a polynomial of degree n with integer coefficients such that $c_n \not\equiv 0 \pmod p$. Prove that the polynomial congruence $f(x) \equiv 0 \pmod p$ has atmost n solutions. 7

   c) Solve simultaneously the following system of congruences :
   $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$. 4

5. a) If n is any integer and p, an odd prime, define the Legendre's symbol $(n|p)$ and prove that $(n|p) \equiv n^{(p-1)/2} \pmod p$. 5

   b) If p and q are distinct odd primes, prove that $(p|q)\,(q|p) = (-1)^{(p-1)(q-1)/4}$. 7

   c) Determine whether 219 is a quadratic residue or nonresidue mod 383. 4

6. a) Prove that there are no primitive roots mod 2^{α}, $\alpha \geq 3$. 4

   b) If p is an odd prime and $\alpha \geq 1$, prove that there exist odd primitive roots g modulo p^{α} and that each such g is also a primitive root modulo $2p^{\alpha}$. 4

   c) If m is not of the form 1, 2, 4, p^{α} or $2p^{\alpha}$, where p is and odd prime, prove that there are no primitive roots mod m. 8

7. a) Define partition of a positive integer n. Give a combinatorial proof of the Euler's identity.

$$1 + \sum_{n=1}^{\infty} \frac{x^{n^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2n})} = \prod_{n=0}^{\infty} (1 + x^{2n+1}).$$ 6

   b) State and prove the Jacobi's triple product identity. 10

8. a) Prove that : $\prod_{n=1}^{\infty} (1-x^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, x^{n(n+1)/2}$. 4

   b) State and prove the Rogers-Ramanujan identities. 12

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